Quantum Limit on Computational Time and Speed

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(July 6, 2001)

Abstract

We investigate if physical laws can impose limit on computational time and speed of a quantum computer built from elementary particles. We show that the product of the speed and the running time of a quantum computer is limited by the type of fundamental interactions present inside the system. This will help us to decide as to what type of interaction should be allowed in building quantum computers in achieving the desired speed.

Any computer, be it classical or quantal, stores information in the physical states and processes the same according to the laws of physics [1]. Last century has witnessed tremendous growth of computational power in solving complex problems and in optimising computational resources. In the saga of information age it is impetuous to know the ultimate limitations on the speed and the maximal running time or the life-time of a computer. Since a computer is a physical device made up of basic elements, its underlying dynamics, its ability, its performance and life time will depend on the laws of physics. The present day computers store information in macroscopic elements and process information according to the laws of classical physics [1]. For example, all the electronic computers employ silicon memory chips and are controlled by electrostatic or electromagnetic interactions. However, the quantum information revolution has offered us possibility of realizing computers that can be built out of a collection of *interacting* two-state systems [2–4]. Using the quantum features such as linear superposition and quantum entanglement, it is hoped that quantum computers can gain some extra power as compared to classical computers [3,4]. For example, one can factorise an integer in polynomial time [5] and search a data base in square root number of steps [6] in a quantum computer. In practical realisation of quantum computers a crucial factor is that the interaction between subsystems should be controlled in such a way that the speed and life time of the computer should be increased. Also the interaction between the computer and external world should be as small as possible to avoid decoherence

[7].

It is often the case that quantum mechanical and relativity principles together with the existence of fundamental constants in nature such as Planck's constant $\hbar$, speed of light c and universal Gravitational constant G some times provide important bound on the physical quantities of interest. For example, it was shown that there is a limit on the maximal acceleration of a quantum particle and accurate clocks are those which can be accelerated maximally [8]. The maximal acceleration was found to be the Planck acceleration $a_p = \sqrt{\frac{\hbar G}{c^3}}$. Similarly, there is a limit on the power radiated by charged particles in external force field [9]. Therefore, it is natural to seek if the ability of a computer is limited by laws of physics, existence of fundamental constants and fundamental forces in the universe. Recently, there have been attempts to put limits on the speed and memory capacity of a computer. Margolus and Levitin [10] have shown that the number of elementary operations that a physical system can perform per second is limited by $\frac{2E}{\pi\hbar}$, where E is average energy of the system. Remarkably, Lloyd [11] has found that the speed of a computer is limited by the energy available to the machine and memory (the total number of bits available for a physical system to process information) is by its entropy. Ng [12] has argued that the product of speed and the clock rate of a computer is limited by the reciprocal of squared Planck time $t_p^2 = c^5/\hbar G$. This bound arises again from laws of quantum physics and gravitation. However, bounds on time and speed has not been shown to depend on the type of fundamental interactions present in the universe.

In recent years one of the important goals is to enhance the speed and the life time (or running time) of a computer. In view of the fact that physical world is ultimately quantum mechanical, there have been momentous developments in imagining quantum computers. To build a quantum computer one needs to know what type of interactions would be most appropriate among its subsystems. In this paper we ask the question: In quantum world, is there a limit on the running time (or life time) and speed of a computer arising due to the fundamental *physical interactions* between the subsystems? In nature, we know that there are foure types of fundamental interactions, namely, (i) strong interaction, (ii) weak interaction, (iii) electromagnetic interaction and (iv) gravitational interaction. Also it is known that fundamental interactions between elementary particles carrying information can be used for quantum logic operations [13–15]. If a quantum computer is built from elementary particles subject to these interactions, then its speed and running time should depend on the type of interaction. By combining principles of quantum theory, relativity and particle physics empirical systematics, we show that the product of running time T and the speed v is upper bounded by $2^n/n$, where n is an integer depending on the type of fundamental interaction present in the quantum system. This can be interpreted as a complimentary relation between speed and time of a quantum computer for a given natural interaction.

First, let us note that quantum computers can be built out of collection of logical two-state systems (called qubits) with suitable physical interactions (from above four fundamental forces) allowed by laws of nature. In principle, we can utilise elementary particles with two distinct quantum states as our qubits. For example, in case of strong interaction, we can identify neutron and proton as two distinct states of a single neucleon with iso-spin interaction. Hence, in the presence of iso-spin interaction a neucleon can be regarded as a qubit. In case of weak interaction, one has to look for elementary particles that are subjected to weak forces. In this case there is a well-known phenomena of neutrino oscillation. By coherence properties [16] one can be regarded it as a qubit. In case of electromagnetic interaction, there are numerous examples. Simplest example, here, is that in the presence of electromagnetic laser pulse, a two-state atom under goes oscillation between its ground and excited states and this can be regarded as a qubit [3]. Or we could think of spin of nuclei being manipulated using magnetic field such as in NMR devices. In case of gravitational interaction, in principle, one can think of a quantum system in a superposition of two distinct mass states and if it is allowed that would constitute a qubit. Once one identifies qubits, then using the desired one-qubit and two-qubits interaction [17] one can construct universal logic gates to manipulate the collection of quantum systems.

Suppose, we have a quantum computer made out of elementary two-state systems with any one of the four fundamental interaction between them. Let the composite quantum system consists of $\mathcal{N}$ elementary subsystems each having mass m. Let M be the rest mass and $\Psi(X)$ be the initial state of the quantum computer. Let us assume that the wavefunction has a linear spread δX at an initial time. By the Heisenberg's uncertainty relation its initial momentum spread will be of the order of $\hbar/\delta X$. Therefore, if the computer is allowed to evolve for a time t, the position of the wavepacket will be given by

$$\delta X(t) \simeq \delta X + \frac{\hbar t}{M \delta X} + (\delta X)_{\text{int}}, \tag{1}$$

where the last term gives the spread due to interaction with the surroundings. We are considering the spread along one dimension but it applies equally well to other dimensions. We wish to operate the quantum computer such that $(\delta X)_{\text{int}}$ is much smaller than the linear spread. To ensure this, it is enough to have an interaction such that the system supports two time-scales, t and ϵt where ϵ is an adiabaticity parameter. In such a situation, the diffusion sets in [18,19] at times of the order of $1/\epsilon$. Thus, our arguments will hold if the decoherence times are much lesser than ϵ^{-1}. In practice, this will always hold. Also, recently it has been shown that the decoherence time (which is same as life time or running time) is $(\Lambda - h_{KS})^{-1}$ for complex (chaotic) systems which decay, where Λ and h_{KS} are respectively the Lyapunov exponent and the Kolmogorov-Sinai entropy of the fractal repeller containing all the periodic orbits residing in the system [20].

The minimum value of the spread occurs for $(\delta X(t))^2 \sim (\delta X)^2 \sim \frac{\hbar t}{M}$. Hence, if the computation takes a time T before the quantum computer decoheres, the resolution one can get for the spread of the wave packet is limited by "standard quantum limit", i.e., $(\delta X)_{SQL} \sim \sqrt{\frac{\hbar T}{M}}$. Therefore, we have

$$\delta X \geq \sqrt{\frac{\hbar T}{M}}. \tag{2}$$

The total time T is also identified as the life time of a computer, because in order to do any useful computation in a quantum computer the system should not decay into some 'unwanted' states. Next, we recall 'Wigner's clock' argument for a general quantum mechanical system. Wigner [21] has argued that if a quantum device can distinguish time-intervals to within an accuracy of τ, then the wave packet of the system will be limited to a length scale $\delta X \leq c\tau$. This means that the wave packet of the quantum computer is limited within the same length scale. By combining the above two, we have the following inequality for these dimensionless quantities

$$\frac{Mc^2T}{\hbar} \geq \frac{T^2}{\tau^2}. \tag{3}$$

Now, we bring another important empirical observation from particle physics. In a series of papers a unifying formalism of elementary particles was developed that remarkably relates their masses and life-times [22–26]. If we have an elementary quantum system with mass m and life time T in presence of certain force field, then it [22–26] has been empirically shown that the dimensionless quantity constructed from m, T, c, and $\hbar$ obey the following relation

$$\frac{mc^2T}{\hbar} = \frac{2^n}{n}, \tag{4}$$

where n is an integer that characterises the *strength and type of physical interactions present.* Multiplying both sides with $\mathcal{N}$ we have $\frac{Mc^2T}{\hbar} = \mathcal{N}\frac{2^n}{n}$, where $\mathcal{N}$ denotes the number of units of mass m placed in making the quantum device. The total mass M of the quantum computer will be given by $M \simeq m\mathcal{N}$. The corrections due to binding energy is neglected here. The above relation has been substantiated for various elementary particles that exists in nature as well as for nuclei [22–26]. For example, using mass-life time relation it was shown that [25] the life time of proton is $\sim 5.33 \times 10^{33}$ years with $n = 225$. This approach is unique because it incorporates two discrete quantities together in a way that is independent of choosing a particular unit of mass and time.

By combining Wigner inequality (3) and the empirical, but exact relation (4) we obtain an inequality

$$\frac{T^2}{\tau^2} \leq \mathcal{N}\frac{2^n}{n}. \tag{5}$$

We notice that $\frac{1}{\tau} = \nu$ is the clock rate of the computer which is same as number of operations per bit per unit time [11,12]. Moreover, T is the total running time of a computer before it decays (in other words for practical purposes before the quantum computer decoheres). $T/\tau = I$ is the maximum number of steps that a computer can sustain in information processing. On defining $v = I\nu$, where v is nothing but the number of logical operations that one can perform per unit time in a quantum computer, it is actually the speed of a computer. Therefore, we have an upper bound on the product of speed and running time of a quantum computer given by

$$v\, T \leq \mathcal{N}\frac{2^n}{n}. \tag{6}$$

Alternatively, we can interpret the above bound by saying that fundamental forces also can set a limit on the number of steps I given by

$$I \leq \sqrt{\mathcal{N}\frac{2^n}{n}}. \tag{7}$$

The most crucial aspect of the above relation is that the rhs of above equation is a function of an integer that characterises the type of interaction present in the system. This suggest that one can label all quantum computer according to their speed, running time and physical interaction and assign an index n. The above relation can be interpreted as a complimentary relation between speed and running time of a quantum computer. For a given interaction, the more the speed is, the less time it takes for doing a computation. It is also clear that more the n is, more can we raise the upper bound for speed for a given running time. Quantum systems subjected to strong interaction correspond to values of n less than 10. If the interaction is of electromagnetic in origin then the value of n lies between 10 and 40. For weak interaction the values of n can be more than 40. However, for gravitational interaction the value of n is not yet established because we still do not have a full quantum theory of gravity nor do we have enough data about particles decaying through gravitational interaction. All present day proposals manipulate quantum states using electromagnetic interaction. If we insert the value of n lying between 10 to 40, our bound suggests that the number of coherent steps that a physical system can sustain with electromagnetic interaction can vary between $I \sim 10\sqrt{\mathcal{N}}$ to $10^6\sqrt{\mathcal{N}}$. Interestingly, in a universe if one makes a quantum computer with protons then the maximum number of steps I goes as $I \sim 10^{36}\sqrt{\mathcal{N}}$. Therefore, proton based quantum computers could process maximum amount of information in principle.

In standard model of quantum computation one assumes that it is possible to do quantum error correction by adjoining additional qubits (for encoding a single qubit in many qubits such as 5-bit code or 9-bit code) [3]. This is done typically to protect quantum information from undergoing decoherence, thus increasing the life time of a computer. Since our arguments hold for any general composite quantum system, it is expected to hold also under the situation when one attaches additional quantum systems. In fact, one can see from (6) that quantum computers that uses additional qubits will enhance the upper bound on speed and time which is indeed expected. Thus our limit holds when one takes into account robust quantum computation and quantum error correction. Further, we will show that our limit gives an upper bound on the quality of coherence of a qubit. If one defines the quality factor of coherence for a qubit as $Q = \pi\nu T$, then it has been known that to surpass active decoherence mechanism we need Q to be larger than $10^4\nu t_{\mathrm{op}}$, where $t_{\mathrm{op}} = 1/v$ is the time taken for an elementary operation [27]. From (6) one can show that the quality factor of a qubit obey the following inequality

$$Q \leq \frac{\pi 2^n}{n}\nu t_{\mathrm{op}}. \tag{8}$$

For example, in the case of electromagnetic interaction Q should be less than (roghly) $10^{12}\nu t_{\mathrm{op}}$. Thus, our limit (8) can tell us what type of interaction can give the desired quality factor for a qubit. It may be worth mentioning that a recent experiment reports [28] that a qubit in a superconducting junction circuit has been designed with a quality factor of quantum coherence $Q \sim 2.5 \times 10^4$. In principle one can go still beyond as suggested here.

Before concluding, we can argue that the entropy of a quantum computer will be limited by the type of interactions present within the system. Recall that [12] I can also be regarded as the amount of information $\mathcal{I}$ that can be registered by the computer (apart from factors like $\ln 2$). Since $S = k \ln 2\mathcal{I}$, where k is Boltzmann constant, using (7) one can show that the entropy of the computer will be limited by $S \sim \sqrt{\mathcal{N}\frac{2^n}{n}}$ Thus depending on the strength of interaction, the number of available states to a quantum system is also limited.

To end with, we have shown that the laws of physics suggest that there exist a limit on the product of speed and running time of a quantum computer. The ultimate limits on the performance of any quantum computer are governed by the physical processes in the universe and we know that all the processes are described by these four fundamental interactions. Our bound is able to grasp this feature succinctly. We hope that this simple, yet novel result based partly on exact and partly on empirical relation, will throw new light in designing future quantum computers built from elementary particles with desired speed and life time. Further, our result suggests that we can assign an index integer 'n' to a class of computers based on its coherence quality that depends on fundamental forces of nature. In addition, following Lloyd [29,30] if we view our universe as a computer then our bound will be important in deciding the running time and speed of the computation that universe is performing.

Acknowledgements: AKP and AM thank H. D. Parab for useful discussions.

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